Coupled Oscillators and Hysteresis in Sparse Networks

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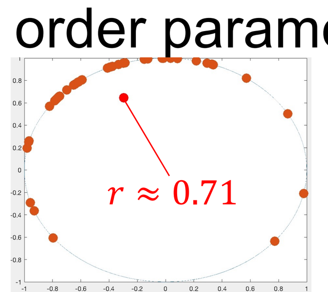
Background

- **Sets of oscillators tend to synchronize when coupled in some manner.**
  - Oscilators modeled by initial frequency ω_i and current phase $\theta_i(t)$.
- **Kuramoto Model:**
  - All N oscillators coupled to all others with strength K.
  - Change of phase θ_i depends on difference in current phase of all other oscillators:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_i - \theta_j)$$

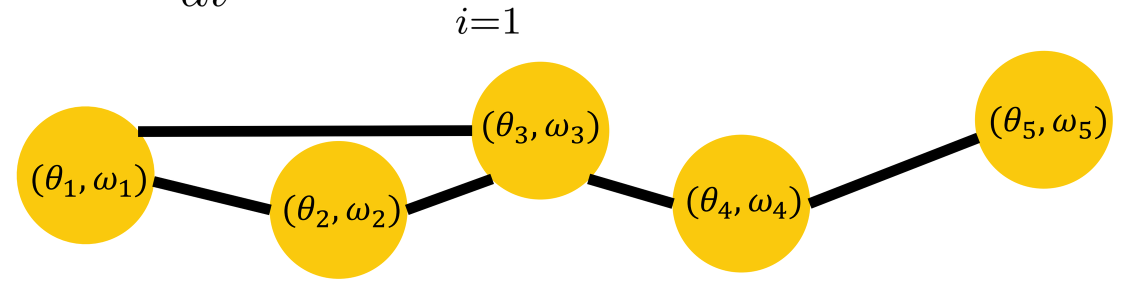
  - Synchrony often measured using order parameter r:

$$r(t) = \left|\frac{1}{N}\sum_{j=1}^{N} e^{i\theta_i(t)}\right|$$



- **Kuramoto Model on Graphs**
  - Arbitrary couplings defined through graph $G = (V, E)$.
    - V is the set of oscillators.
    - E is the set of couplings between oscillators.
  - Change of phase θ_i depends on phase of neighbors:

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{i=1}^{N} K a_{ij}\sin(\theta_i - \theta_j)$$



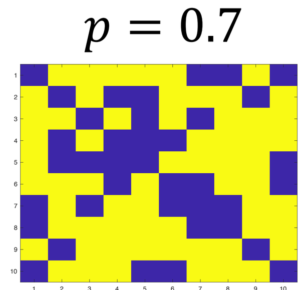
Implementation

- **Order parameter r only for connected graphs**
  - We only consider networks with adjacency matrix A such that:

$$A(p) = [a_{i,j}] \text{ s.t. } p = \frac{1}{N^2}\sum_{i,j=1}^{N} a_{i,j}$$

$$a_{i,i+1} = a_{i+1,i} = 1, a_{i,i} = 0 \quad \forall i$$

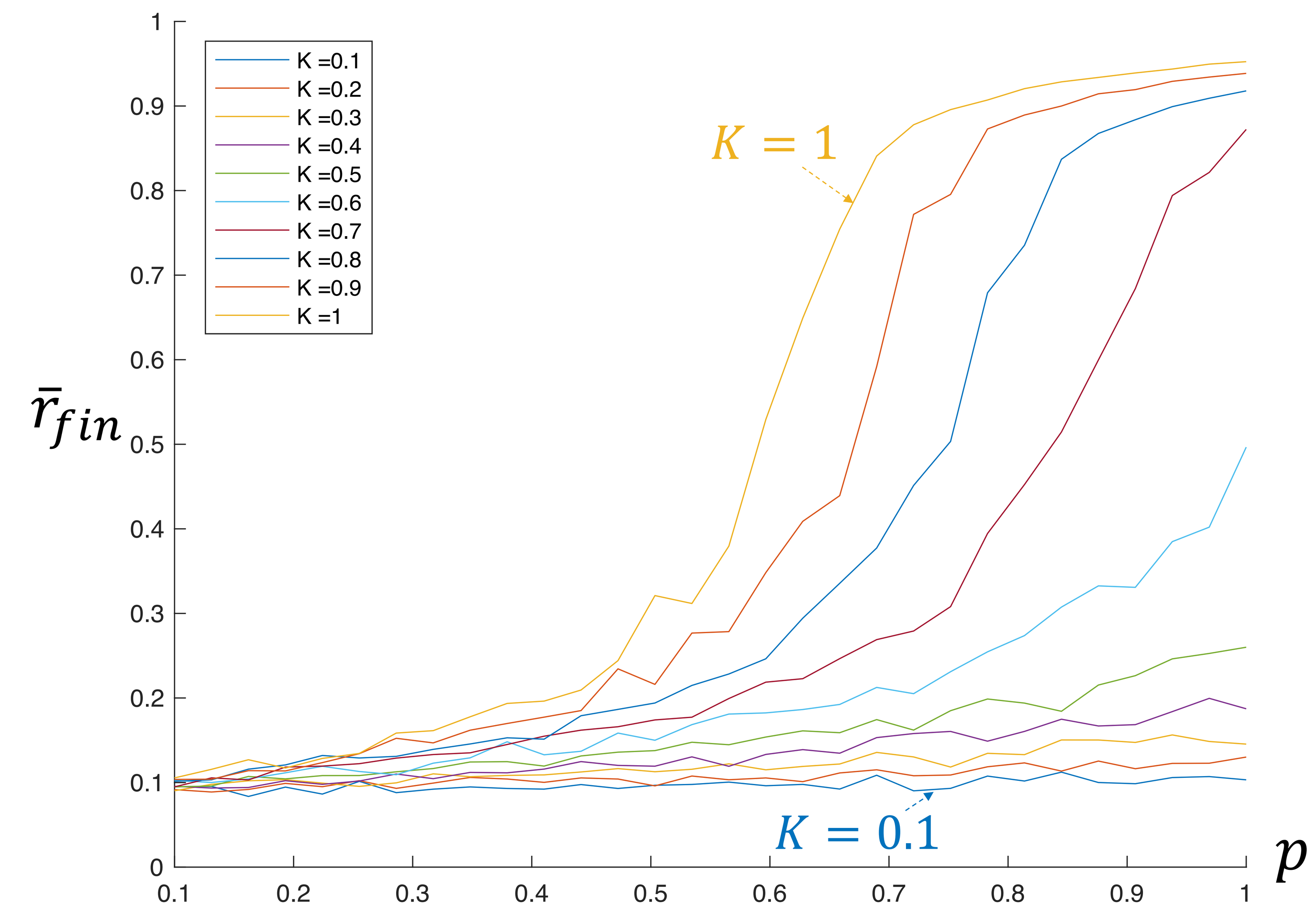
ex: $N = 50$, $p = 0.7$



  - Level of graph connectivity varied by p.
- **Initial conditions for solving IVP:**
  - Normally distributed frequencies $\{\omega_i\} \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, 0.1)$
  - Uniformly distributed initial phase $\{\theta_i(0)\} \overset{\text{i.i.d}}{\sim} \text{Unif}(0, 2\pi)$
- **Runge - Kutta 4th order ODE solver,** $dt = 0.1$

Results: Static K and p

- **Order parameter at 1000 steps recorded for each (K, p).**
  - **500 Oscillators, Results averaged over 10 trials**



Conclusions

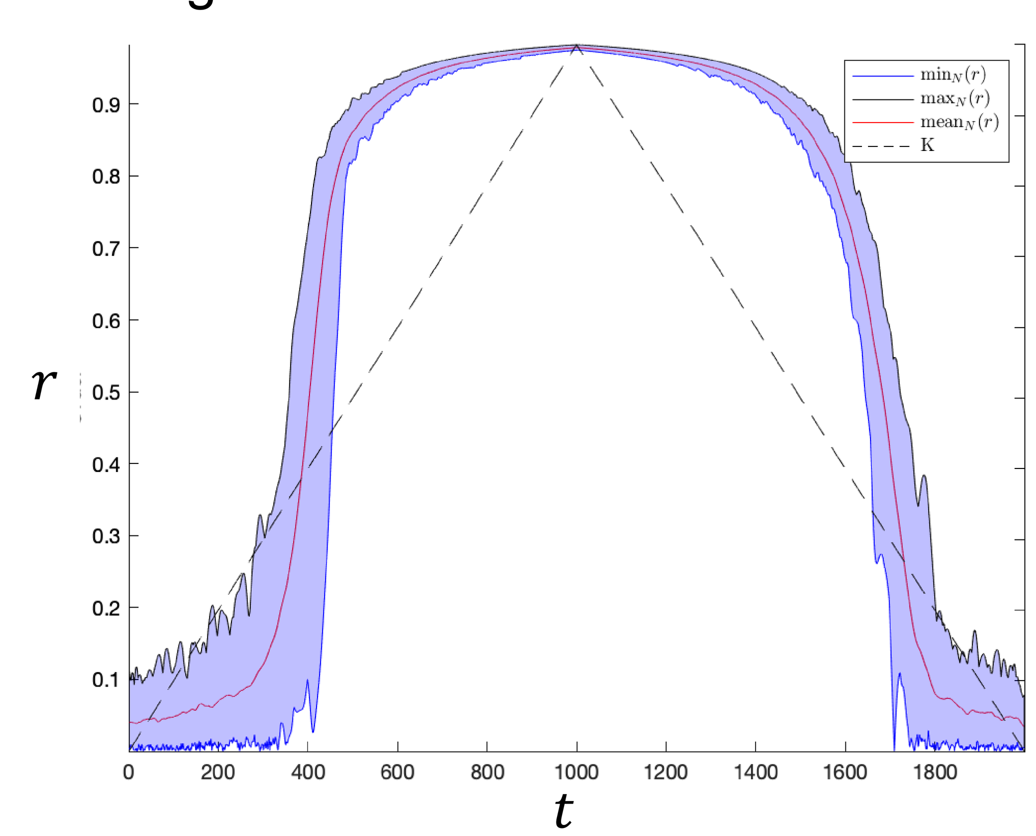
- **For static coupling K and connectivity p:**
  - Sigmoidal relationship between synchronization ability and network connectivity p for each coupling K.
- **For time varying coupling K(t) and static connectivity p:**
  - Hysteresis increases or decreases with p depending on the choice of coupling function $K(t)$.
  - Connectivity is not a hysteresis operator on r depending on K and initial state: heavily rate dependent.

Future work

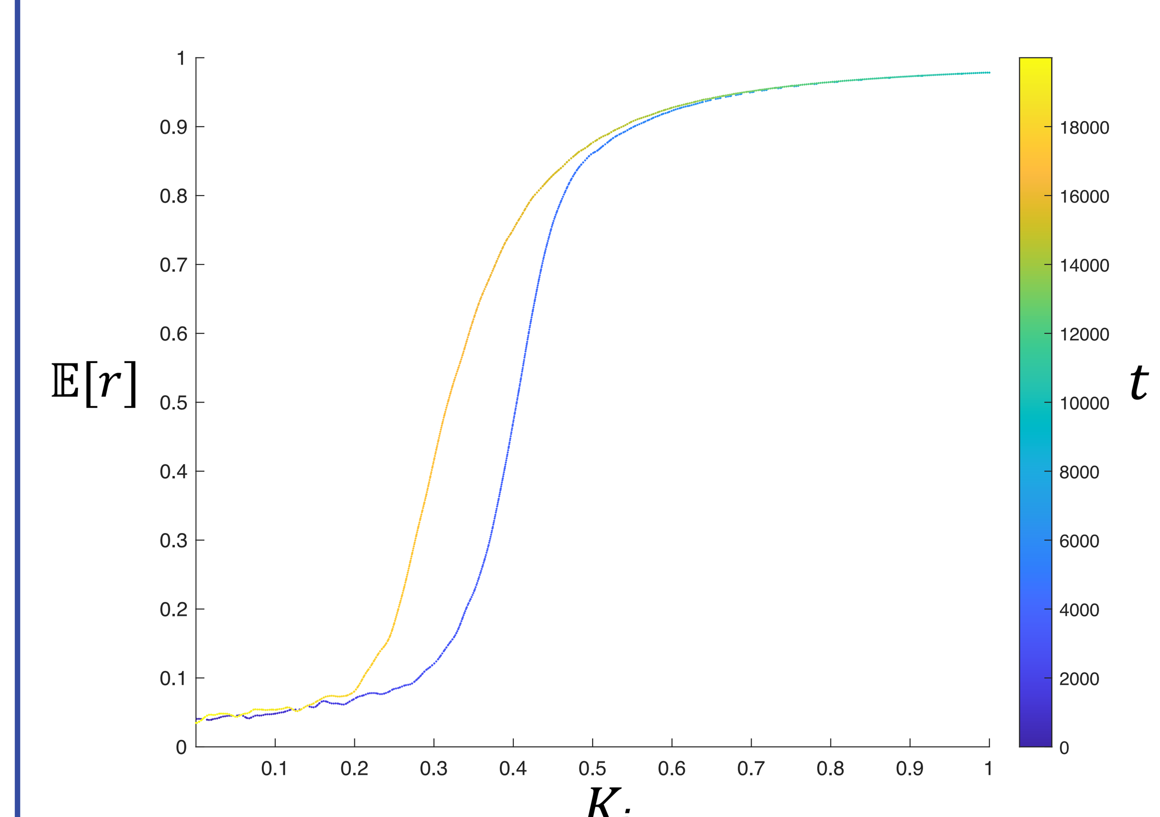
- Why does hysteresis occur?
- What determines the size of this hysteresis effect?
- Why does hysteresis grow/shrink with connectivity?
  - What property of $K(t)$ determines the direction?
- Verify results for other constructions of connected graphs.

Results: Time Varying $K(t)$, Static p

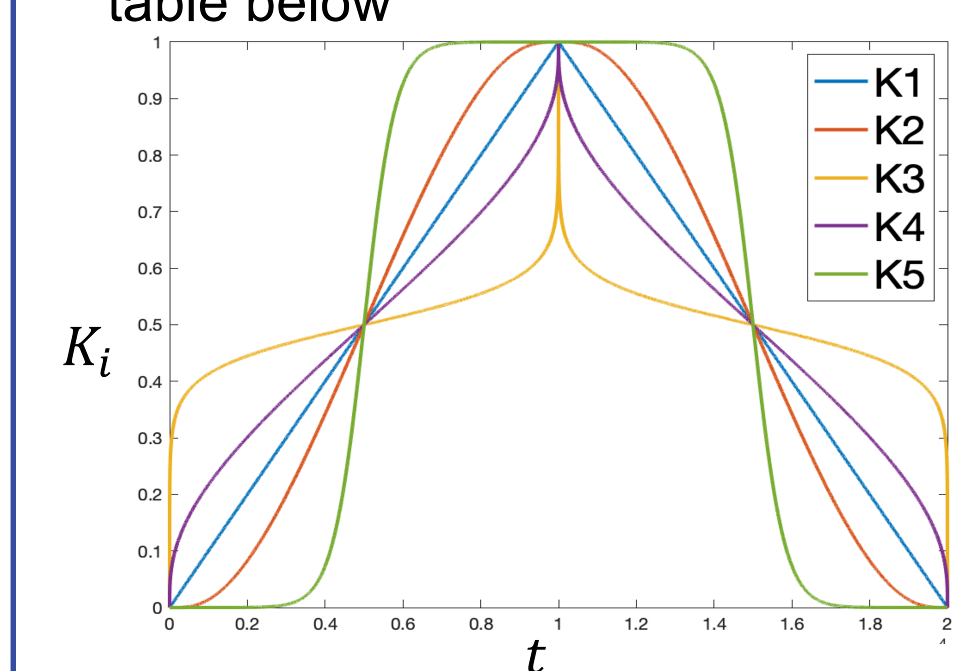
- **Synchrony $r(t)$ measured over time while the coupling $K_i(t)$ is varied.**
- Shown: r(t) range over 100 trials for 500 oscillators at p = 0.50 with K_1 - a triangular function.



- **Hysteresis effect seen as $\mathbb{E}[r(t)]$ is plotted as a function of $K_i(t)$. [$\mathbb{E}$ is over 100 trials]**
- Shown: $\mathbb{E}[r(t)](K)$ 500 oscillators at p = 0.50 with triangular K_1.



- Shown: K_i referenced in hysteresis table below



p:	0.25	0.50	0.75	1.00
K_1	0.033	0.032	0.030	0.030
K_2	0.043	0.045	0.041	0.039
K_3	0.047	0.151	0.242	0.262
K_4	0.028	0.029	0.035	0.045
K_5	0.128	0.139	0.116	0.092

Highlighted: highest (red) in row.
Values normalized by number of steps T.

- **Hysteresis effect for each (p, K_i) quantified by numerically finding the area between the two curves. K_i given by figure above table. 50 Oscillators with $\mathbb{E}[r]$ on 100 trials.**
- All K_i bijective and monotonic increasing/decreasing on $(0, \frac{T}{2})$ / $(\frac{T}{2}, T)$ respectively, with boundary values $||K_i|| = \frac{T}{2}$, $K_i(0) = K_i(T) = 0$, $K_i\left(\frac{T}{2}\right) = 1$.